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Array Sidelobes, Error Tolerance, Gain, and Beamwidth

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In this report, we pres... a set of universal probability curves that relate the array error tolerance, illumination function, achievable sidelobe level, and the designed sidelobe level. We show that this set of probability curves can be used to find a lower designed sidelobe level to achieve a better error tolerance. The relations of array directivity gain and beamwidth with respect to illumination function are also discussed. These parameters can be conveniently used for this design procedure without detailed knowledge of the array illumination.

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Array Sidelobes, Error Tolerance, Gain, and Beamwidth

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CONTENTS

ARRAY SIDELOBES, ERROR TOLERANCE, GAIN, AND BEAMWIDTH

INTRODUCTION

Directivity, beamwidth, and sidelobe levels are the three most important parameters of a phased array. These parameters are determined by the system requirement. However, they are related to each other. These three parameters determine the cost of a phased array. However, another important factor concerning cost is the array error tolerance. Particularly in recent years the requirements of jamming proof and clutter rejection radar lead to a need of extremely low sidelobe array antennas. Theoretically, one can design an array with any sidelobe level as one desires. However, the array errors limit the actual sidelobe level one can achieve. Naturally, the lower the sidelobe level one wishes to achieve the tighter is the array error required and the cost of the array increases accordingly. This brings up the question of whether one may design an array with much lower sidelobe level than required and then let the sidelobe level deteriorate to achieve a better array error tolerance. However, as is well known, as soon as one brings down the sidelobe level the directivity and beamwidth are all affected. To keep the constant directivity and beamwidth, the array size must be increased. Hence, it is a trade-off of increase in array size to achieve a better error tolerance. It would be extremely useful if some relations between these factors can be established.

Many papers have been published on the effect of the random errors in the performance of array antennas. Ruze [1] published the first work on this subject and pursued the effect of small random phase and amplitude errors on the average sidelobe level. His results showed that the effect of the distribution error is to add an essentially constant power level proportional to the mean square error to the mean sidelobe. Individual arrays and particular spatial directions show sidelobe radiation differing from this constant value. Bailin and Ehrlich [2] considered the problem of a linear standing-wave fed slot array and investigated the effect in the radiation pattern of a Gaussian distribution of errors in slot length and slot positions. Their results cannot be used for a general case. Gilbert and Morgan [3] treated the effect on gain and sidelobe level of random geometrical errors in a general two-dimensional aperture. Their results were again in terms of average sidelobe similar to Ruze's conclusion. Elliott [4] has extended the problem to a two-dimensional scanning dipole array subject to random mechanical and electrical excitation errors. He has shown the sidelobe increase due to random errors does not depend upon scan angle. However, his results were also in terms of expected sidelobe level.

One may notice that most of the previous works treated the effect of random errors on the average sidelobe level. The reason to adopt this approach is that the average sidelobe is a deterministic function while the individual sidelobe level is a probabilistic function. In a modern radar system, the peak sidelobe level, however, is more important. Operation requirements are usually specified in terms of peak sidelobe level, in particular for a low sidelobe system. The reason is that for jamming rejection, it is most important that a jammer noise comes from a spatial location at the peak of a sidelobe and this noise must not interfere with the radar detection. Average sidelobe level in this case does not help. An array may have a very low average sidelobe level, but at the same time, it may have a few high peak sidelobes located at the wrong spot, which renders the radar system vulnerable.

In this report we relate this peak sidelobe level as function of the array errors. We also present a set of design curves which enable a designer to choose a design sidelobe level to achieve a desired

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sidelobe with an acceptable array error tolerance. Often in the system design stage, detailed array design data such as illumination function are not available. However, parameters such as directivity and beamwidth, which are directly related to the system requirement, are known. It would be advantageous if these parameters can be used to estimate the array error tolerance without the knowledge of the array detailed design data. A set of design curves for such a purpose is presented, and examples are given.

THE STATISTICS OF RADIATION PATTERNS OF AN ARRAY WITH ERRORS

The array pattern of a planar array can be represented by

$$G(\mu,\nu) = \sum_{nm}\sum A_{nm} \exp\left[j(m\mu + n\nu)\right] \tag{1}$$

where

$$\mu = 2\,\pi\, d_x/\lambda\ (\sin\theta\cos\phi - \sin\theta_0\cos\phi_0),$$

$$\nu = 2\,\pi\, d_y/\lambda\ (\sin\theta\sin\phi - \sin\theta_0\sin\phi_0),$$

and d_x and d_y are the array element spacings, respectively, in the x and y directions while λ is the wavelength. Angles θ_0 and ϕ_0 are the beam point direction and θ and ϕ are the direction of a field point.

Let us assume that each array element is subject to an independent and random error. For convenience, we shall separate this error into two parts; the amplitude error δ and the phase error ϕ [5]. The above equation then becomes

$$G(\mu,\nu) = \sum_{nm}\sum A_{nm}(1+\delta_{nm}) \exp\left(j\,\phi_{nm}\right) \exp\left[j(m\mu + n\nu)\right]. \tag{2}$$

When the array illumination is symmetrical and no error is present, the array pattern function $G(\mu,\nu)$ becomes a sum of cosine functions. Therefore the pattern function of Eq. (2) is similar to the case that was analyzed by Rice [6] in which a steady sinusoidal current is added with a random noise. In the present case, it is slightly different in the sense that separate amplitude and phase errors are added to each cosine term. However, it can be shown that the probability density function of $G(\mu,\nu)$ is similar to the case treated by Rice and it has a Rician distribution in the form [5,7]

$$p(R) = \frac{R}{\sigma^2} \exp\left[-(R^2 + \bar{g}_1^2)/2\sigma^2\right] I_0\left(\frac{R\bar{g}_1}{\sigma^2}\right) \tag{3}$$

where $R = G(\mu,\nu)$ and $\bar{g}_1$ is the real part of the expected value of $G(\mu,\nu)$. When the array is symmetrically illuminated and the phase error is small, it can be shown that $\bar{g}_1$ is approximately equal to the ideal (design) array pattern function. Quantity σ is the standard deviation of the array error which can be approximated in the following form. When array error δ and ϕ are small (see the appendix of this report) [7,8]

$$\sigma^2 \approx \frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2)\sum_{nm}\sum A_{nm}^2 \tag{4}$$

where σ_δ and σ_ϕ are the rms values of the amplitude error (measured in fraction of A_{nm}) and phase error (measured in radians) respectively. The amplitude error is also assumed to have zero mean. For better comparison, it is also assumed that the illumination function A_{nm} is normalized such that

$$\sum_{nm}\sum A_{nm} = 1. \tag{5}$$

In other words, the main beam has an amplitude of unity or 0 dB. All sidelobe levels and σ values in later discussions are referred to this level. The function $I_0(x)$ is the modified Bessel function of zero order.

Equation (3) relates the actual sidelobe level (R) and the array σ as a function of $\bar{g}_1(\mu,\nu)$ which is equal to the design sidelobe level. One may normalize both R and σ with respect to the function $\bar{g}_1$ and achieve a universal function which is independent of the design sidelobe level. If this is done, one has the following cumulative probability function:

$$P(S < S_L) = \int_0^{S_L} \frac{S}{\sigma'^2} \exp\left(-\frac{S^2+1}{2\sigma'^2}\right) I_0\left(\frac{S}{\sigma'^2}\right) dS \qquad (6)$$

where S is the normalized sidelobe level in the sidelobe region and σ' is the normalized σ. Both S and σ' are normalized with respect to the design sidelobe.

Figure 1 is a family of such curves with σ' as the parameter. Each of these curves presents the cumulative probability that S (in terms of design sidelobe) is less than or equal to a level S_L for a given σ'. Since this curve is presented in such a way that it is not a function of the angle (μ,ν), the curve applies to all points in the sidelobe region. Secondly, the curve is normalized with respect to the design sidelobe level. It represents the probability of the deviation of sidelobe level from the design value. Although the curves are not presented explicity as a function of (μ,ν), they are related to the sidelobe level. For example, at the peak of a sidelobe, the normalized σ' may be only equal to 0.1, but at a point where the sidelobe level is ten times smaller, the normalized σ' then becomes 10 times larger. One can see the difference in the probability distribution for these two cases. This set of curves is universal. It applies to arrays with different illumination designs, different sizes, and different errors.

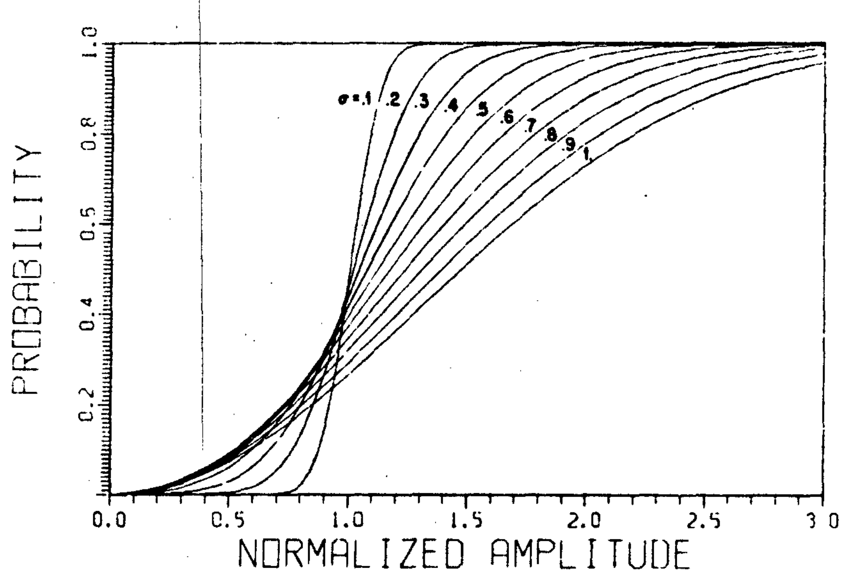

Fig. 1 — Cumulative probability of sidelobe level with different array error standard deviation σ' (both σ' and sidelobe level are normalized with respect to the designed sidelobe level)

With the aid of this plot, one may easily determine the required error tolerance to achieve a desired sidelobe level. For example, one may wish to design an array having a 50 dB sidelobe, with probability of 90% that the sidelobe level will not exceed the designed level by more than 30%. The curve for $\sigma' = 0.2$ satisfies this condition, because at $S = 1.3$ the cumulative probability is 90%. Since $\sigma' = 0.2$ is equivalent to -14 dB, the required σ^2 is approximately -64 dB (50 + 14).

CONSTANT PROBABILITY OF SIDELOBE LEVEL

Although Fig. 1 contains all necessary design information, it is not in a very convenient form to use. In most cases, a designer would like to know what kind of error budget one has to have in order to maintain peak sidelobes not to exceed a certain limit. A curve that directly relates these peak sidelobe levels and array errors would be most useful. With slight modifications of Fig. 1, a set of constant probability contour curves is presented in Fig. 2, which is tailored for this purpose. These curves are plotted for given cumulative probabilities from $P = 0.9$ to 0.98 (Eq. (6)). The ratio of the design sidelobe level to the desired sidelobe level (in dB scale) is used as the abcissa. The desired sidelobe is the actual sidelobe one can achieve even if one designs the array illumination function to achieve the design sidelobe level. The product of the square of the sum of the rms phase and amplitude errors and a factor D is used as the ordinate. The factor D is defined as

$$D = \sum_{nm}\sum A_{nm}^2/\text{Desired Sidelobe Level.} \tag{7}$$

As an example, suppose we design a linear Chebyshev array of 100 elements at a 40 dB peak sidelobe and we would like to keep the array peak sidelobe no more than −37 dB with a probability of 0.9 after it is built. The illumination function $\sum_n A_n^2 = -19$ dB. Therefore, the factor D is $D = 37 - 19 = 18$ dB. We find from Fig. 2 that

$$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2) \cdot D \approx -14 \text{ dB}$$

or

$$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2) \approx -32 \text{ dB.}$$

If we assume that the amplitude error and phase error are equal, then

$\sigma_\delta = 0.025$

$\sigma_\phi = 1.44$ deg.

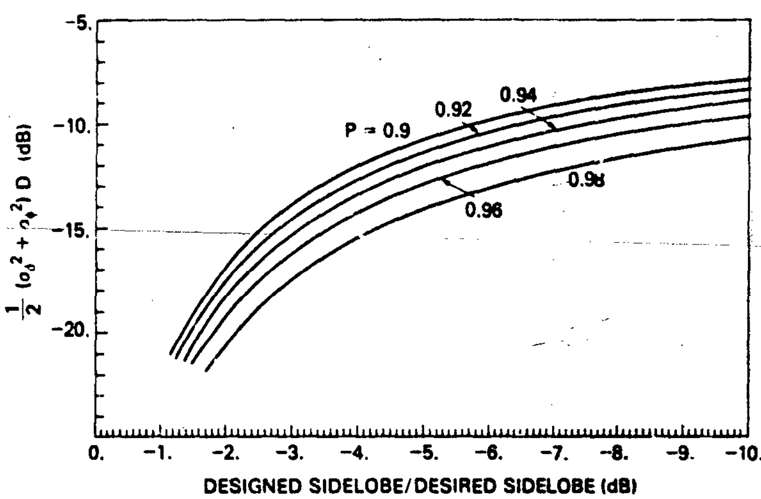

Fig. 2 — Constant probability contour; σ_δ and σ_ϕ are respectively rms amplitude and phase errors; $D = \Sigma A_{nm}^2$/desired sidelobe where A_{nm}s are array illuminations

In general, it is difficult to achieve a smaller error. One may, therefore, attempt to design an array with lower sidelobe level and let it deteriorate more in order to achieve a better array error tolerance yet at the same time to maintain the desired sidelobe level. For instance in the previous example, if one designs a −45 dB array intead of −40 dB, to achieve the same −37 dB sidelobe, one has from Fig. 2:

$$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2)D = -8.7 \text{ dB}.$$

Since the designed sidelobe is changed, the illumination function is also changed and

$$\sum A_n^2 = -18.8 \text{ dB}$$

and

$$D = -18.8 + 37 = 18 \text{ dB}$$

$$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2) = -26.9 \text{ dB}$$

or

$$\sigma_\delta = 0.0452$$

$$\sigma_\phi = 2.6 \text{ deg},$$

which has a larger error tolerance than the previous example.

The above examples are summarized in Table 1.

Table 1

Probability	0.9	0.9
Designed Sidelobe (dB)	−40	−45
Desired Sidelobe (dB)	−37	−37
Difference (dB)	3	8
$\sum A_{nm}^2$ (dB)	−19	−18.8
D (dB)	18	18
$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2)D$ (dB)	−14	−8.7
$\frac{1}{2}(\sigma_\delta^2 + \sigma_\phi^2)$ (dB)	−32	−26.9
σ_ϕ (deg)	1.44	2.6
σ_δ	0.025	0.0452

The patterns of the above two examples are plotted in Fig. 3 for the −40 dB and −45 designs. The phase and amplitude errors of each element of these two examples are generated by the random number generator built in the computer with a Gaussian distribution of the above prescribed σ values of phase and amplitude errors. The patterns of this array with these random generated errors and that of the same array without errors are plotted in Figs. 3(a) and 3(b). Comparing the idea pattern and the one with errors, one may see that out of these 50 peak sidelobes, three or four of them slightly exceed the −37 dB limit, which is about a 10% probability as we predicted. These examples support the validity of this approximation.

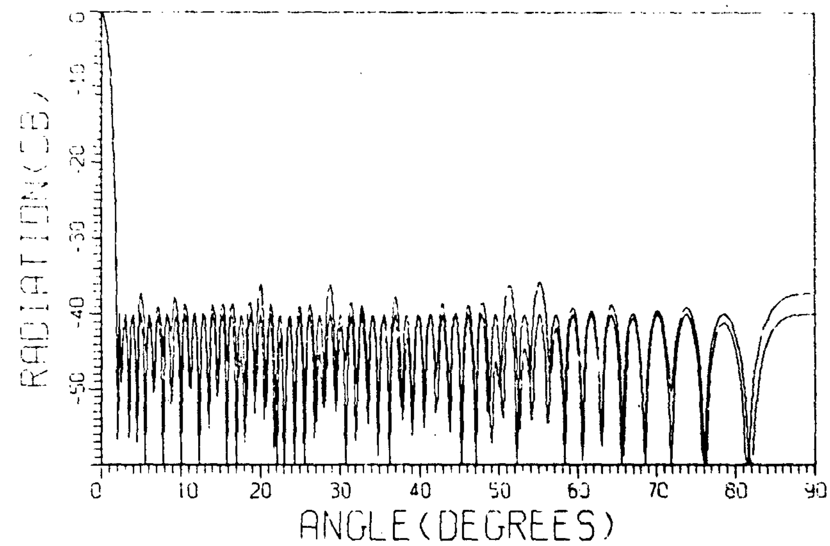

Fig. 3(a) — 40 dB Chebyshev array pattern and pattern with error designed for 0.9 probability at −37 dB; $\sigma_\delta = 0.025$, $\sigma_\phi = 1.44°$

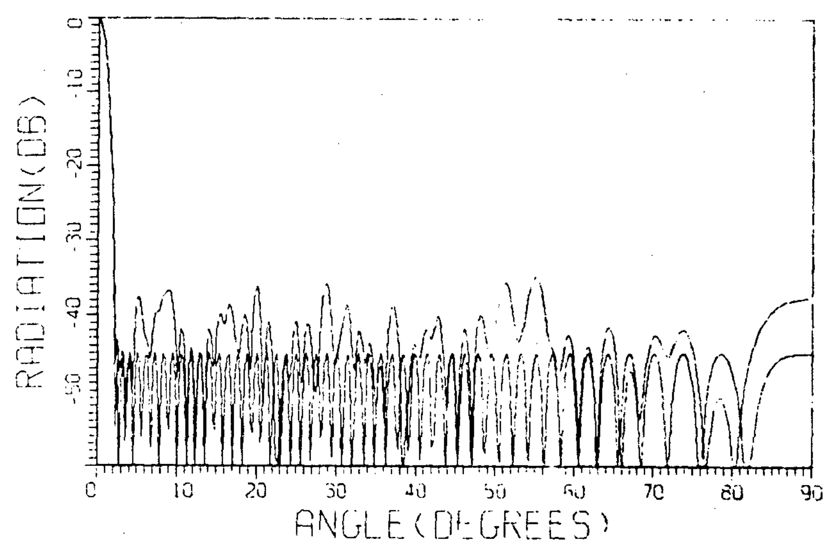

Fig. 3(b) — 45 dB Chebyshev array pattern and pattern with error designed for 0.9 probability at −37 dB; $\sigma_\delta = 0.0452$, $\sigma_\phi = 2.1°$

The probabilities of peak sidelobes to exceed −37 dB in both cases are the same. However, one may notice that the radiation pattern of the case designed at −45 dB is worse in the sense that many sidelobes are smeared while in the case of −40 dB design, the sidelobes are merely slightly perturbed. This may be one of the prices one has to pay to achieve a higher array error tolerance. Furthermore, when a lower sidelobe illumination is used, in general, the array main beam tends to become fatter and directivity gain reduced. In the next section, this effect is discussed further.

DIRECTIVITY GAIN AND BEAMWIDTH

The above design procedure requires a knowledge of the array illumination function. This parameter may not be available for a system engineer at the system design level, he may not be ready to commit himself to certain illumination functions at the time being. Therefore, it would be more convenient if there are other parameters that he can use and which are specified at the system level. In the this section we show that the array directivity or beamwidth can be used in place of illumination function for this design purpose.

Elliott [9] showed that the directivity of a planar array is:

$$D_g = 4\,\pi\, d_x\, d_y \cos\theta_0 / \sum_{nm}\sum A_{nm}^2 \tag{9}$$

where

$$A_{nm} = A_n\, A_m,$$

and A_n and A_m are respectively the row and column illumination of the array. The angle θ_0 is the beam scan angle from the array normal when the directivity gain is measured, and d_x and d_y are the element spacings in terms of wavelength. The product of d_x and d_y represent the array grid area. To avoid supergain and grating lobes, most arrays are designed with element spacings which are equal or slightly greater than half wavelength depending on the scan range. Hence,

$$4\, d_x\, d_y = A_g \geq 1.$$

The array illumination function can be represented by

$$\sum_{nm}\sum A_{nm}^2 = \pi\, A_g \cos\theta_0 / D_g.$$

This relation can be used to replace the array illumination function. A set of constant probability curves in Fig. 2 is replotted and shown in Fig. 4. The abcissa of this plot is the same as that shown in Fig. 2 which is the ratio of the designed sidelobe level to the desired sidelobe level in dB scale. The ordinate of this plot, however, is in terms of $(\sigma_\delta^2 + \sigma_\phi^2)/F$, and the constant factor F is defined as

$$F = \frac{2 D_g\, S_b}{\pi\, A_g \cos\theta_0} \tag{10}$$

where S_b is the desired sidelobe level. This set of curves is more convenient to use if the directivity gain is known. For example, if the array is specified to have a sidelobe level better than −40 dB at a probability of 0.9, the array directivity gain is specified to be 40 dB at the broadside. Assume that the array grid is half wavelength, then

$$A_g = 1$$

$$D_g S_b = 1$$

$$F = 0.6366.$$

Suppose that we assume that the ratio of the designed sidelobe to the desired sidelobe is set at −3 dB. From Fig. 4, we find that

$$(\sigma_\delta^2 + \sigma_\phi^2)/F = 0.042.$$

$$\sigma_\delta^2 + \sigma_\phi^2 = 0.042 \times 0.6366.$$

Assume

$$\sigma_\delta = \sigma_\phi.$$

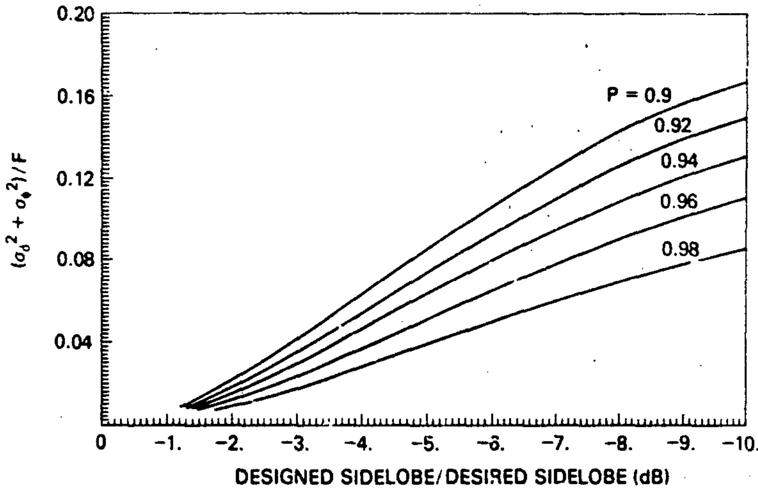

Fig. 4 — Constant probability contour; σ_δ and σ_ϕ are respectively rms amplitude and phase errors. $F_3 = 2D_g S_o/\pi \; A_g \cos\theta_0$ (see Eq. (9) for details)

Then the required amplitude error is

$$\sigma_\delta = 0.1156$$

and phase error is

$$\sigma_\phi = 6.6247 \text{ deg.}$$

For the case that the ratio of designed and desired sidelobe level is set at −8 dB, the amplitude error becomes

$$\sigma_\delta = 0.21$$

and phase error becomes

$$\sigma_\phi = 12 \text{ deg.}$$

The above example is summarized in Table 2:

Table 2

Probability	0.9	0.9
Desired Sidelobe (dB)	−40	−40
Designed Sidelobe (dB)	−43	−48
Difference (dB)	−3	−8
Directivity Gain D_g (dB)	−40	−40
A_g	1	1
$\cos\theta_0$	1	1
F_3	0.6366	0.6366
$(\sigma_\delta^2 + \sigma_\phi^2)/F_3$	0.042	0.144
σ_δ	0.1156	0.21
δ_ϕ (deg)	6.63	12.3

Elliott [9] also showed that the directivity gain and the beamwidth of a planar is related by the relation

$$D_g = 32400/B \tag{11}$$

where B is the aerial 3 dB beamwidth in square degrees measured at the same scan angle θ_0 where the directivity gain D_g is measured. Since it is more convenient to use dB scale, Eq. (11) can be converted to the form

$$D_g = 45.1 - B \text{ (dB)}. \tag{12}$$

For example, if the antenna has a 3.24 square degrees beamwidth measured at broadside, then $B = 5.1$ dB and the directivity gain would be 40 dB. This relation can be used in Eq. (9) to determine the corresponding amplitude and phase errors.

For a linear array, the directivity gain can be approximated by the relation

$$D_g = \sum_n A_n^2 \tag{13}$$

if the array spacing is in the order of half wavelength. The beamwidth of a linear array is related to the directivity gain by the relation

$$D_g = 101.5/\theta_0 \tag{14}$$

where θ_0 is the linear array 3 dB beamwidth.

These two relations can be used in place of the array illumination function as the examples shown in the planar array case. Curves shown in Fig. 4 hence can be applied directly to a linear array case.

Figure 4 shows that the array error tolerance is directly proportional to the ratio of the design sidelobe level to the desired sidelobe level. For a fixed beamwidth or directivity, the array tolerance becomes better as the sidelobe ratio increases. However, one should notice that when the design sidelobe is reduced to maintain a constant beamwidth or directivity, the array aperture size must be increased. In effect, it is a trade-off of a larger aperture for a better error tolerance. An optimal point of best cost-effectiveness may be achieved. However, it depends on each individual system and array antenna types.

CONCLUSIONS

The statistical distribution of sidelobe level of an array antenna due to random phase and amplitude errors is Rician. The derivation of its associated parameters of this distribution and their relations to the array element errors are shown in the appendix. These results are then used to represent a set of universal probability curves that relates the array error tolerance, illumination function, achievable sidelobe level, and the designed sidelobe level. Constant probability curves for design convenience are presented. Examples are given that show how these curves can be used to achieve a better error tolerance yet maintain a desired sidelobe level. Array directivity gain and beamwidth, which are usually defined at the system design level, are shown how they can be used in the design of array error tolerances and sidelobe level.

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APPENDIX

Define a complex random variable

$$G = x + jy \tag{A1}$$

$$E\{|G - E(G)|^2\} = \sigma_x^2 + \sigma_y^2 \tag{A2}$$

where

$$\sigma_x^2 = E\{[x - E(x)]^2\} \tag{A3}$$

$$\sigma_y^2 = E\{y - E(y)]^2\} \tag{A4}$$

$$E\{[G - E(G)]^2\} = \sigma_x^2 - \sigma_y^2 + 2j\,\sigma_{xy} \tag{A5}$$

$$\sigma_{xy} = E\{[x - E(x)][y - E(y)]\} \tag{A6}$$

$$G(u,v) = \sum_{nm}\sum A_{nm}\,(1 + \delta_{nm})\exp(j\,\phi_{nm}) \cdot \exp[j(m\mu + n\nu)] \tag{A7}$$

$$E[G] = \sum_{nm}\sum \Phi(1) A_{nm} \exp[j(m\mu + n\nu)] \tag{A8}$$

where

$$\Phi(k) = \int p(\phi)\exp(jk\phi)\,d\phi \tag{A9}$$

is the characteristic function of random variable ϕ. When the phase error ϕ is small, Φ (1) is very close to unity, hence $E(G)$ is equal to the array pattern shown in Eq. (1). When the array is symmetrically illuminated $E(G)$ becomes real.

$$|E[G]|^2 = \sum_{nmrs}\sum\sum\sum |\Phi(1)|^2 A_{nm} A_{rs} \exp[j(m-s)\mu + j(n-r)\nu] \tag{A10}$$

$$|G|^2 = \sum_{nmrs}\sum\sum\sum A_{nm} A_{rs}(1 + \delta_{nm})(1 + \delta_{rs})$$

$$\cdot \exp[j(\phi_{nm} - \phi_{rs})] \cdot \exp[j(m-s)\mu + j(n-r)\nu] \tag{A11}$$

$$E[|G|^2] = \sum_{nm}\sum A_{nm}^2(1 + \sigma_\delta^2) + \sum_{nmrs}\sum\sum\sum A_{nm} A_{rs} |\Phi(1)|^2$$

$$m \neq s \text{ or } n \neq r$$

$$\cdot \exp[j(m - s)\mu + j(n - r)\nu] \tag{A12}$$

where σ_δ^2 is the variance of the amplitude error δ.

$$\begin{aligned}\sigma_x^2 + \sigma_y^2 &= E[|G - E(G)|^2] \\ &= E(|G|^2) - |E(G)|^2 \\ &= \sum_{nm}\sum A_{nm}^2[1 + \sigma_\delta^2 - |\Phi(1)|^2]\end{aligned} \tag{A13}$$

$$(G)^2 = \sum\sum\sum\sum_{nmrs} A_{nm} A_{rs} (1 + \delta_{nm})(1 + \delta_{rs})$$

$$\cdot \exp [j(\phi_{nm} + \phi_{rs})] \exp [j(m + s)\mu + j(n + r)\nu] \tag{A14}$$

$$E [G]^2 = \sum\sum\sum\sum_{nmrs} A_{nm} A_{rs} \Phi^2(1) \exp [j(m + s)\mu + j(n + r)\nu]$$

$$n \neq r \text{ or } m \neq s$$

$$+ \sum\sum_{nm} (1 + \sigma_\delta^2)\Phi\ (2)\ A_{nm}^2 \exp [j(2m\mu + 2n\nu)] \tag{A15}$$

$$[E\ (G)]^2 = \sum\sum\sum\sum_{nmrs} A_{nm} A_{rs} \Phi^2(1) \exp [j(m+s)\mu + j(n+r)\nu] \tag{A16}$$

$$\sigma_x^2 - \sigma_y^2 = \mathrm{Re}\{E\ [(G)^2] - [E\ (G)]^2\}$$

$$= \sum\sum_{nm} [(1 + \sigma_\delta^2)\Phi(2) - \Phi^2(1)]\ A_{nm}^2 \cos (2m\mu + 2n\nu) \tag{A17}$$

$$\sigma_{xy} = \frac{1}{2}\mathrm{IM}\ \{E\ (G)^2] - [E\ (G)]^2\}$$

$$= \frac{1}{2}\sum\sum_{nm} [(1 + \sigma_\delta^2)\Phi\ (2) - \Phi\ (1)]\ A_{nm}^2 \sin (2m\mu + 2n\nu)]. \tag{A18}$$

Solving for σ_x and σ_y from Eqs. (A3) and (A17), one finds

$$\sigma_x^2 = \frac{1}{2}\sum\sum_{nm} A_{nm}^2 [C + D \cos (2m\mu + 2n\nu)] \tag{A19}$$

$$\sigma_y^2 = \frac{1}{2}\sum\sum_{nm} A_{nm}^2 [C - D \cos (2m\mu + 2n\nu)] \tag{A20}$$

where

$$C = 1 + \sigma_\delta^2 - |\Phi(1)|^2 \tag{A21}$$

$$D = (1 + \sigma_\delta^2)\Phi\ (2) - \Phi^2(1), \tag{A22}$$

since

$$\Phi\ (k) = \int p(x) \exp [jk\phi] dx.$$

If the phase error is limited to a small value, the above equation can be approximated;

$$\Phi(k) = \int p(\phi)\left[1 + jk\phi - \frac{k^2\phi^2}{2} + \ldots\right] dx$$

$$\Phi(1) \approx 1 - \frac{\overline{\phi^2}}{2} + j\overline{\phi}$$

$$|\Phi(1)|^2 \approx 1 - \overline{\phi^2} + \overline{(\phi)}^2$$

$$\approx 1 - \sigma_\phi^2.$$

$$C \approx \sigma_\delta^2 + \sigma_\phi^2 \tag{A23}$$

$$\phi(2) \approx 1 - 2\overline{\phi^2} + j2\overline{\phi}$$

$$(\phi(1))^2 \approx 1 - \overline{\phi^2} - \overline{(\phi)}^2 + 2j\overline{\phi}$$

$$D \approx \sigma_\delta^2 - \sigma_\phi^2. \tag{A24}$$

In the above derivation all terms that are higher than 2nd order have been neglected.

One may notice that when the phase error and amplitude error are in the same order, then $D \approx 0$. Furthermore if

$$2\,m\mu + 2\,n\nu \neq k\pi,$$

the term $D \cos (2\,m\mu + 2\,n\nu)$ is much smaller than C (see Eqs. (A19) and (A20)); under these conditions

$$\sigma_\xi^2 \approx \sigma_\eta^2 \approx \frac{1}{2} \sum_n \sum_m A_{nm}^2 (\sigma_\delta^2 + \sigma_\phi^2) \qquad \text{(A25)}$$

and if the array is symmetrically illuminated,

$$\sigma_{\xi\eta} = 0.$$

The probability density function of random variable $G(\mu, \nu)$ hence has a Rician distribution as shown in Eq. (3).

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